

# Canonical $SO(2,4)$ -invariant quantization in conformally flat spaces

Sofiane Faci\*

*Centro Brasileiro de Pesquisas Físicas (CBPF-ICRA), Rua Dr. Xavier Sigaud,  
150, Urca, CEP 22290-180, Rio de Janeiro, RJ, Brasil*

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We show how to quantize  $SO(2,d)$ -invariant fields in  $d > 2$  dimensional conformally flat spaces (CFS). The Weyl equivalence between CFSs is exploited to perform the quantization process in Minkowski space then transport the entire  $SO(2,d)$ -invariant structure to curved CFSs. We make use of the canonical quantization scheme and a special careful is made to specify a scalar product, technically related to a Cauchy surface. The latter is chosen to be common to all globally hyperbolic CFSs in order to relate the different associated Hilbert spaces. The quantum fields are constructed and the two-point functions are given in terms of their minkowskian counterparts. It appears that an  $SO(2,d)$ -invariant quantum field does not locally distinguish between two different CFSs.

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## I. INTRODUCTION

Conformal invariance is a fundamental ingredient for modern theoretical physics and covers several areas, see for instance [1] and references therein. There is a rich literature dealing with Weyl invariance [2–5] and also with  $SO(2,d)$ -invariant field theories (CFT) in  $d > 2$  spaces [6–11]. The case of  $d = 2$  is particular since the conformal group becomes infinite [12] and, exploring the Polyakov formalism, was at the origin of String Theory [13]. On the other hand, the set of conformally flat spaces (CFS) is of great importance. Indeed it includes Minkowski, FLRW and de Sitter spaces. According to the standard cosmological model, our Universe has been in a de Sitter phase during inflation and has afterwards gone through an expanding - currently accelerating - FLRW space, it might come back to a de Sitter shape in the far future [14, 15]. Moreover, CFSs represent the first step toward general curved manifolds. Thus, well understand the quantization process in these spaces is a capital step towards quantum field theory in general curved spaces and quantum gravity.

Weyl and the restricted conformal group  $SO(2,d)$  transformations are different but subtilely related. Indeed, a  $d > 2$  dimensional classical Weyl invariant field theory restricted to live in a CFS yields an  $SO(2,d)$ -invariant field theory [16]. The demonstration of this result relies on the  $SO(2,4)$ -invariant structure transport of a classical field theory from Minkowski space to a CFS. The present paper extends this approach to include quantum fields. The main goal is to construct standard  $SO(2,d)$ -invariant quantum field theories in arbitrary CFSs.

For this purpose, we use the canonical quantization scheme which goes by three steps: i) Construct a Hilbert space ii) Define a unitary representation iii) Find a causal reproducing kernel.

A Hilbert space is a complete set of normalized modes, relative to a scalar product which is technically related to a Cauchy surface. The key point of the present work is that it is possible to choose the same Cauchy surface for all CFSs [17]. This allows to have a one-to-one correspondance between all Hilbert spaces in the different CFSs and consequently to simply relate the Wightman functions. The idea is to explore the Weyl equivalence between CFSs to perform the quantization process in Minkowski space then transport the entire  $SO(2,d)$ -invariant structure to curved CFSs. This is possible since the  $SO(2,4)$  is the smallest group containing as subgroups all isometry groups associated to the CFSs [18]. Also, an  $SO(2,4)$ -invariant field theory in a CFS is also invariant under the associated isometry group, which means a well defined theory.

To develop, let  $\mathcal{M}$  be a real  $d > 2$  dimensional differential manifold. Minkowski space reads  $(\mathcal{M}, \eta_{\mu\nu})$  where  $\eta_{\mu\nu} = (+, - - \dots)$ . An arbitrary CFS  $(\mathcal{M}, \bar{g}_{\mu\nu})$  is locally related to Minkowski space by a Weyl rescaling

$$\bar{g}_{\mu\nu} = K^2 \eta_{\mu\nu}, \quad (1)$$

where the Weyl factor  $K$  is a real, non vanishing  $C^\infty$  function. The considered spaces are globally hyperbolic, otherwise Cauchy problem is not well defined. In such spaces, the present formalism can be applied to bounded subspaces where a Cauchy surface can be defined. Our methodology is based on two maps: the map  $W$  to transport the fields and the map  $H$  to transport the differential operators acting on these fields. That is, for a giving conformal field  $F$  and an arbitrary operator  $\mathcal{O}$ , defined in Minkowski space, the two maps read

$$W : F \rightarrow \bar{F} = WF \quad (2)$$

$$H : \mathcal{O} \rightarrow \bar{\mathcal{O}} = W\mathcal{O}W^{-1}, \quad (3)$$

where  $W$  denotes the map and its matrix representation. The map (2) is usually defined as  $\bar{F} = K^s F$  (where  $s$  is called the conformal weight) but can be extended to include a tensorial part. Say, for a tensor field with components  $F_I$ , the general form of the matrix  $W$  reads

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\*Electronic address: sofiane@cbpf.br

$\bar{F}_I = K^s (\delta_I^J + \Upsilon_I^J) F_J$ , where  $\Upsilon_I^J$  is a non-diagonal matrix which depends on the function  $K$  and its derivatives. Explicit examples of such matrices can be found in [19, 20]. As a consequence, the field  $\bar{F}$  and the operator  $\bar{\mathcal{O}}$  are defined in the CFS  $(\mathcal{M}, \bar{g}_{\mu\nu})$ .

## II. CANONICAL QUANTIZATION IN MINKOWSKI SPACE

### A. A classical theory

Let us start by considering a classical free field  $F$  defined in Minkowski space and obeying to the  $SO(2, 4)$ -invariant equation

$$\mathcal{E}F = 0, \quad (4)$$

where  $\mathcal{E}$  is a given differential operator. Furthermore we assume that this equation does not need boundary conditions - other than initial data on some Cauchy surface - to be solved. The  $SO(2, 4)$ -invariance means

$$\forall e \in SO(2, 4), \quad \mathcal{E} U_e F(x) = w_e(x) \mathcal{E} F(x), \quad (5)$$

where

$$U_e F(x) = M_e(x) F(e^{-1}.x), \quad (6)$$

where  $w_e(x)$  and  $M_e(x)$  are matrices. This representation can also be expressed in an infinitesimal form by the commutation relations between the group generators  $X_e \in so(2, 4)$  and the field  $F$

$$[X_e, F] = X_e F + \Sigma_e F, \quad (7)$$

where  $X_e F$  denotes the scalar action and  $\Sigma_e F$  the tensorial action. The infinitesimal form of (5) reads

$$\forall X_e \in so(2, 4), \quad [\mathcal{E}, X_e] = \xi_e \mathcal{E}, \quad (8)$$

where  $\xi_e$  are real functions. Since  $SO(2, 4)$  is a Lie group, the finite transformations are obtained from the infinitesimal ones by the exponential application,

$$U_e F(x) = e^{\lambda_e^i X_e^i} F(x). \quad (9)$$

### B. A quantum theory

The  $SO(2, 4)$ -covariant quantization is achieved by constructing a Hilbert space - on which  $SO(2, 4)$  acts unitarily - and an invariant Wightman function.

A Hilbert space is a complet set of normalized modes  $\mathcal{H} = \{F_k\}$ . These are solutions of Eq. (4) and normalized,

$$\langle F_k, F_{k'} \rangle = \delta_{kk'}, \quad (10)$$

according to a scalar product  $\langle, \rangle$ . To ensure unitarily the scalar product should be  $SO(2, 4)$ -invariant, that is

$$\forall e \in SO(2, 4), \quad \langle U_e F_k, U_e F_{k'} \rangle = \langle F_k, F_{k'} \rangle. \quad (11)$$

The Wightman two-point function reads

$$\mathcal{W}(x, x') = \sum_k F_k^*(x) F_k(x'), \quad (12)$$

and provides a causal and covariant reproducing kernel of  $\mathcal{H}$ :

- A reproducing kernel.

$$\forall F \in \mathcal{H}, \quad \langle \mathcal{W}(x, \cdot), F \rangle = F(x). \quad (13)$$

- Causality.

$$\mathcal{W}(x, x') = \mathcal{W}(x', x) \quad (14)$$

as soon as  $x$  and  $x'$  are causally separated.

- $\mathcal{W}$  is  $SO(2, 4)$ -invariant. This comes from the relations (11), (12) and (13).

Afterwards the quantum field  $\hat{F}$  is constructed as

$$\hat{F}(x) = \hat{a}(\mathcal{W}(x, \cdot)) + \hat{a}^\dagger(\mathcal{W}(x, \cdot)), \quad (15)$$

where  $\hat{a}$  and  $\hat{a}^\dagger$  are respectively anti-linear and linear operators - actually, they are operator-valued tempered distributions - acting on a Fock space.

As a consequence, using (12), the above field can be expanded as

$$\hat{F}(x) = \sum_k F_k(x) \hat{a}_k + F_k^*(x) \hat{a}_k^\dagger, \quad (16)$$

where  $\hat{a}_k = \hat{a}(F_k)$  and  $\hat{a}_k^\dagger = \hat{a}^\dagger(F_k)$  are the standard annihilation and creation operators of the modes  $F_k$  and obeying the canonical commutation relations (ccr)

$$\begin{aligned} [\hat{a}_k, \hat{a}_{k'}] &= [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0, \\ [\hat{a}_k, \hat{a}_{k'}^\dagger] &= \epsilon_k \delta_{kk'}, \end{aligned} \quad (17)$$

where  $\epsilon_k = \pm 1$ . In the case  $\epsilon_k$  does not have the same sign for all modes  $F_k$ , the space  $\mathcal{K}$  is then of a Krein type. Note that the appearance of the non-trivial algebraic structure indicates the entrance into the quantum world.

The one particle sector of the Fock space, denoted  $\underline{\mathcal{H}}$ , contains the quantum states

$$|F_k\rangle = \hat{a}^\dagger(F_k) |0\rangle, \quad (18)$$

where the conformal vacuum state  $|0\rangle$  verifies

$$\forall \hat{a}_k, \quad \hat{a}_k |0\rangle = 0, \quad (19)$$

and is the unique invariant state of  $\underline{\mathcal{H}}$ ,

$$\forall e \in SO(2, 4), \quad \underline{U}_e |0\rangle = |0\rangle. \quad (20)$$

So the  $SO(2, 4)$ -invariance permits to pick up a preferred vacuum state, which corresponds to a preferred basis of

$\mathcal{H} = \{F_k\}$ . This solves one of the most persistent difficulties in QFT in curved spaces.

The Wightman function (12) can thus be written under the usual form

$$\mathcal{W}(x, x') = \langle 0 | \hat{F}(x) \hat{F}(x') | 0 \rangle. \quad (21)$$

The resulting quantum field has the following properties.

- It verifies the field equation

$$\forall F_1, F_2 \in \mathcal{H}, \quad \langle F_1 | \mathcal{E} \hat{F} | F_2 \rangle = 0. \quad (22)$$

For more details see the appendix of [19].

- Is causal. If  $x$  and  $x'$  are causally disjoint then

$$[\hat{F}(x), \hat{F}(x')] = 0. \quad (23)$$

- Is  $SO(2, 4)$ -covariant

$$\forall e \in SO(2, 4), \quad \underline{U}_e^{-1} F(x) \underline{U}_e = F(e.x). \quad (24)$$

### III. GOING TO A CFS

#### A. Transporting the classical theory

The two maps  $W$  and  $H$  allow to transport the equation (4) to the CFS. The new equation reads

$$\begin{aligned} \bar{\mathcal{E}} \bar{F} &= H \mathcal{E} W F \\ &= W \mathcal{E} W^{-1} W F \\ &= W(\mathcal{E} F) \\ &= 0. \end{aligned} \quad (25)$$

The CFS generators of  $so(2, 4)$  result from applying the map  $H$  on  $\{X_e\}$ :

$$X_e \rightarrow \bar{X}_e = H X_e = W X_e W^{-1}. \quad (26)$$

They act on the field  $\bar{F}$  as

$$\begin{aligned} [\bar{X}_e, \bar{F}] &= \bar{X}_e \bar{F} + \bar{\Sigma}_e \bar{F} \\ &= W X_e F + \Sigma_e W F \\ &= W [X, F], \end{aligned} \quad (27)$$

where  $\bar{\Sigma}_e = \Sigma_e$  because the spinorial action  $\Sigma_e$  does not contain derivatives. Yet the finite transformations are obtained by the exponential application,

$$\begin{aligned} \bar{U}_e \bar{F}(x) &= e^{\lambda_e^i \bar{X}_e^i} \bar{F}(x), \\ &= W(x) e^{\lambda_e^i X_e^i} F(x), \\ &= W(x) U_e F(x), \\ &= W(x) M_e(x) F(e^{-1}.x), \\ &= W(x) M_e(x) [W(e^{-1}.x)]^{-1} \bar{F}(e^{-1}.x), \\ &= \bar{M}_e(x) \bar{F}(e^{-1}.x). \end{aligned} \quad (28)$$

The Minkowskian and CFS representations are related through  $\bar{M}_e(x) = W(x) M_e(x) [W(e^{-1}.x)]^{-1}$ .

The invariance of Eq. (25) follows from the third line of (28) and yields,  $\forall e \in SO(2, 4)$ ,

$$\begin{aligned} \bar{\mathcal{E}} \bar{U}_e \bar{F}(x) &= W \mathcal{E} U_e F(x), \\ &= W w_e(x) \mathcal{E} F(x), \\ &= w_e(x) \bar{\mathcal{E}} \bar{F}(x). \end{aligned} \quad (29)$$

The infinitesimal invariance is easier to implement

$$\forall X_e \in so(2, 4), \quad [\bar{\mathcal{E}}, \bar{X}_e] = W \zeta_e \mathcal{E} W^{-1} = \zeta_e \bar{\mathcal{E}}. \quad (30)$$

Note that the identity  $\bar{\zeta}_e = \zeta_e$ , or equivalently  $\bar{w}_e = w_e$ , means that the resulting CFS theory keeps the same  $SO(2, 4)$ -invariant structure of the minkowskian theory.

#### B. Transporting the quantum theory

Let us turn to the quantum field. The modes  $\{F_k\}$ , solutions of (4), are transported using the map  $W$  to get the modes

$$\bar{F}_k = W F_k, \quad (31)$$

solutions of the CFS equation (25). The set of these new modes  $\{\bar{F}_k\}$  forms a basis for a new Hilbert space denoted  $\bar{\mathcal{H}}$ . This space is equipped with a new scalar product  $\langle\langle, \rangle\rangle$ , defined in such a way to ensure the normalization of the modes  $\bar{F}_k$  in the same way as for the modes  $F_k$ :

$$\langle\langle \bar{F}_k, \bar{F}_{k'} \rangle\rangle = \langle F_k, F_{k'} \rangle = \epsilon_k \delta_{kk'}. \quad (32)$$

This is possible since we are free to choose the same Cauchy surface to define both minkowskian and CFS scalar products [17]. Moreover the scalar product  $\langle\langle, \rangle\rangle$  is  $SO(2, 4)$ -invariant,

$$\langle\langle U_e \bar{F}_k, U_e \bar{F}_{k'} \rangle\rangle = \langle\langle \bar{F}_k, \bar{F}_{k'} \rangle\rangle, \quad (33)$$

which means that here again  $U$  acts unitarily on  $\bar{\mathcal{H}}$ .

The Wightman two-point function reads

$$\begin{aligned} \bar{\mathcal{W}}(x, x') &= \sum_k \bar{F}_k^*(x) \bar{F}_k(x'), \\ &= W(x) W(x') \mathcal{W}(x, x'), \end{aligned} \quad (34)$$

which automatically provides a causal and covariant reproducing kernel of  $\bar{\mathcal{H}}$ :

- A reproducing kernel,

$$\forall \bar{F} \in \bar{\mathcal{H}}, \quad \langle\langle \bar{\mathcal{W}}(x, \cdot), \bar{F} \rangle\rangle = \bar{F}(x). \quad (35)$$

- Causality. It comes from that of  $\mathcal{W}$  and the fact that a Weyl rescaling preserve the space causal structure. That is

$$\bar{\mathcal{W}}(x, x') = \bar{\mathcal{W}}(x', x) \quad (36)$$

as soon as  $x$  and  $x'$  are causally separated.

- Finally,  $\bar{\mathcal{W}}$  is  $SO(2,4)$ -invariant. This comes from the unitarity condition (33).

The CFS quantum field is then constructed with the new Wightman function,

$$\hat{\bar{F}}(x) = \hat{a}(\bar{\mathcal{W}}(x, .)) + \hat{a}^\dagger(\bar{\mathcal{W}}(x, .)), \quad (37)$$

which yields

$$\bar{\mathcal{W}}(x, x') = \langle \bar{0} | \hat{\bar{F}}(x) \hat{\bar{F}}(x') | \bar{0} \rangle. \quad (38)$$

The vacuum state  $|\bar{0}\rangle$  is the conformal one. It is the unique invariant state of  $\underline{\mathcal{H}}$ ,

$$\forall e \in SO(2,4), \quad \underline{\bar{U}}_e |\bar{0}\rangle = |\bar{0}\rangle. \quad (39)$$

As a consequence, using (12), the above field can be expanded as

$$\hat{\bar{F}}(x) = \sum_k \bar{F}_k(x) \hat{a}_k + \bar{F}_k^*(x) \hat{a}_k^\dagger, \quad (40)$$

where the annihilators and creators are formally identical to the Minkowskian ones, verifying the algebra (17), but acting on the modes  $\{\bar{F}_k\}$ :  $\hat{a}_k = \hat{a}(\bar{F}_k)$ ,  $\hat{a}_k^\dagger = \hat{a}^\dagger(\bar{F}_k)$  and thus verifying  $\hat{a}_k |\bar{0}\rangle = 0$ . The one particle sector of the CFS Fock space  $\underline{\mathcal{H}}$  contains the quantum states

$$|\bar{F}_k\rangle = \hat{a}^\dagger(\bar{F}_k) |\bar{0}\rangle. \quad (41)$$

The resulting quantum field

- Verifies the field equation

$$\forall \bar{F}_1, \bar{F}_2 \in \bar{\mathcal{H}}, \quad \langle \bar{F}_1 | \bar{\mathcal{E}} \hat{\bar{F}} | \bar{F}_2 \rangle = 0. \quad (42)$$

- Causal. If  $x$  and  $x'$  are causally disjoint then

$$[\hat{\bar{F}}(x), \hat{\bar{F}}(x')] = 0. \quad (43)$$

- $SO(2,4)$ -covariant

$$\forall e \in SO(2,4), \quad \underline{\bar{U}}_e^{-1} \bar{F}(x) \underline{\bar{U}}_e = \bar{F}(e.x). \quad (44)$$

Note that the quantum fields  $\hat{F}$  and  $\hat{\bar{F}}$ , nor the quantum states  $|F_k\rangle \in \underline{\mathcal{H}}$  and  $|\bar{F}_k\rangle \in \underline{\bar{\mathcal{H}}}$ , are not explicitly

related. Though this does not prevent the Wightman functions to be related through

$$\bar{\mathcal{W}}(x, x') = W(x)W(x') \mathcal{W}(x, x'). \quad (45)$$

This is the crux of the present paper. This is well known for the scalar field but not in the general case that was demonstrated here.

#### IV. CONCLUDING REMARKS

We have succeeded in formally transport the whole  $SO(2,4)$ -invariant structure of a minkowskian quantum field theory to obtain a new  $SO(2,4)$ -invariant theory defined in an arbitrary conformally flat space. Two key points were explored. i) It is possible to choose the same Cauchy surface in all CFSs in order to define both minkowskian and curved CFS scalar products. ii) The  $SO(2,4)$  group is the smallest group containing as subgroups all isometry groups associated to the CFSs.

The  $SO(2,4)$ -invariant quantum structure in two CFSs relies on the same quantum operators acting on different but one-to-one related Hilbert spaces. It appears that an  $SO(2,4)$ -invariant quantum field does not locally distinguish between two different CFSs. This is particularly true for high-frequency modes which only "interact" with the close neighborhood.

Note that the Weyl rescaling (1) and thus the maps (2) and (3) are local and depend on a coordinate system, which does not, in general, cover the whole spaces. Nevertheless the  $SO(2,4)$  is a Lie group and the global action can be obtained from the infinitesimal one using the exponential application. Moreover, in case of need, several coordinate systems can be used to cover the whole spaces (examples are given in [21, 22]).

Note also that only free fields were considered in this work, also no interaction-like conformal anomalies could appear [23]. Nonetheless free-like conformal anomalies can appear in bounded curved spaces - like de Sitter space - [24, 25]. These are important and problematic global effects that come from the gravitational interaction feedback. The present work focused on local properties and did not treat conformal anomalies.

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